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Study of Gravitomagnetic Clock Effect Due To Gravitational Spin-Orbit Coupling

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Abstract: ***In this paper, we consider a possible gravitational spin-orbit coupling. It is shown that in presence of such a coupling there would appear a clock effect very similar to the gravitomagnetic clock effect. The new clock effect is found to be topological. According to this effect, two counter orbiting spinning test particles placed on two identical circular equatorial orbits around a central massive body would take different times to complete a full revolution. In this paper, we calculate the period difference using the gravitational spin-orbit coupling term as given by the expression found by Barker and O'Connel.***

KEYWORDS: *Gravitational spin-orbit coupling, Gravitomagnetic Clock Effect, Schwarzschild and Kerr Effect.*

INTRODUCTION

General relativity predicts that two freely counter-revolving test particles in the exterior field of a central body take different periods is usually termed as the gravitomagnetic clock effect [1]. The general relativistic derivation of this effect for a general orbit is quit complicated. The gravitomagnetic clock effect arises from the differences in orbital period in prograde and retrograde orbits. Prograde orbit is one where the sense of orbital motion of the test body is same as the sense of rotation of the central body. In retrograde orbit the situation is opposite. The period difference is $4\pi a/c$, which is the expression for gravitomagnetic clock effect. Clearly, the retrograde orbit is faster than the prograde orbit.

Geometrical aspects of gravitomagnetic clock effect have been studied by Tartaglia [2]. Influence of cosmological constant on g'clock effect is studied in [3] and influence of cosmic strings is studied in [4]. A brief discussion of this effect in case the bodies are of comparable mass can be found in [5].

When a non-spinning test particle orbits a Kerr source, there appears the gravitomagnetic clock effect which is the difference between orbital periods of the test particle motion in prograde and retrogradeorbits. When a spinning particle orbits a spin less central object, the case is that of motion of a spinning particle in Schwarzschild metric. The clock effect in this case is also termed as gravitomagnetic clock effect [1, 6, and 7].

The gravitomagnetic clock effect is the asymmetry in the arrival time of a pair of intersecting time-like geodesic, stemming from the same source and having opposite azimuthal angular momentum [8]. In this paper, the treatment of spinning test particle in Schwarzschild space gives the result $T_+ - T_- = 6\pi S/mc^2$, where T_+ is the period of orbital motion of the spinning particle in a co-rotating orbit and T_- is the same in counter-rotating orbit. Here, co-rotating orbit has positive variation of azimuthal angular coordinate with respect to a fixed sense of a static observer. This calculation is analogous to the treatment of Bini et al [8].

Thus Eq. (29) is the gravitomagnetic clock effect in spinning test particle orbit around a Schwarzschild source. One thing to note here is that the contribution of spin of the orbiting test particle is always given by Eq. (29), no matter whether it is in motion round a Schwarzschild source or a Kerr source. Therefore, the source of asymmetry in period of prograde and retrograde motion of a spinning particle is the spin orbit coupling.

The gravitational spin-orbit coupling is analogous to the hydrogenicspin-orbit coupling. There is only the kinematical Thomas precession in the accelerated frame while the gravitational spin-orbit coupling as well as the Thomas precession [9] exists in the Schwarzschild frame. The gravitational spin-orbit coupling seems to violate EEP [10]. The Thomas precession is purely kinematical in origin. The hydrogenic spin-orbit couplingalso has the same form as the electromagnetic Thomas precession, but the origin is quite different. The origin of the hydrogenic spin-orbit coupling is the Ampere's law. In the rest frame of the electron, the moving proton generates a magnetic field according to Ampere's law and this magnetic field interacts with the electron spin. Thus the origin of the gravitational spin-orbit coupling is the gravitational Ampere's law that is a moving mass generates a coupling to the spin. In the rest frame of a Dirac particle, the gravitational mass M moves and it generates the spin-gravity coupling according to the gravitational Ampere's law [10].

This paper is organized as follows: In section 2, we have discussed about gravitomagnetic clock effect. In section 3, a short calculation of clock effect due to gravitational spin-orbit coupling is given. Finally in section 4, we give a brief summary.

1. GRAVITOMAGNETIC CLOCK EFFECT

The gravitomagnetic clock effect arises in orbit of a particle which is moving round a Kerr source of mass M on the equatorial plane. The Kerr metric for the equatorial plane$\theta = {}^{\pi}/_{2}$, dθ=0, is given by

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 - 2a\left(\frac{2M}{r}\right)dtd\varphi + \frac{1}{1-\frac{2M}{r}+\frac{a^2}{r^2}}dr^2 + \left(\frac{2M}{r}a^2 + r^2 + a^2\right)d\varphi^2, \qquad (1)$$

Where units are such that G=c=1. Throughout the paper we shall use this convention unless otherwise specified. The motion of test particle of mass m in the Kerr field is governed by the geodesic equation:

$$\frac{d^2x^{\varpi}}{ds^2} + \Gamma^{\mu}_{\alpha\beta}\frac{dx^{\alpha}}{ds}\frac{dx^{\beta}}{ds} = 0, \qquad (2)$$

where$\Gamma^{\mu}_{\alpha\beta}$ are Christoffel symbols.

To evaluate the gravitomagnetic clock effect, we need only the geodesic equation for the radial coordinate r:

$$\frac{d^2r}{ds^2} + \Gamma^r_{tt}\left(\frac{dr}{ds}\right)^2 + \Gamma^r_{rr}\left(\frac{dr}{ds}\right)^2 + \Gamma^r_{\varphi\varphi}\left(\frac{d\varphi}{ds}\right)^2 + \Gamma^r_{t\varphi}\frac{dt}{ds}\frac{d\varphi}{ds} + \Gamma^r_{\varphi t}\frac{d\varphi}{ds}\frac{dt}{ds} = 0. \qquad (3)$$

Now,

$$\Gamma^r_{tt} = \frac{1}{2}g^{rm}\left(\frac{\partial g_{mt}}{\partial t} + \frac{\partial g_{mt}}{\partial t} - \frac{\partial g_{tt}}{\partial x^m}\right) = g^{rr}\frac{\partial g_{rt}}{\partial t} - \frac{1}{2}g^{rr}\frac{\partial g_{tt}}{\partial r} = -\frac{1}{2}g^{rr}g_{tt,r}, \qquad (4)$$

Since $g_{rt} = 0$.

Similarly,

$$\Gamma^r_{rr} = \frac{1}{2}g^{rr}g_{rr,r'}\Gamma^r_{\varphi\varphi} = -\frac{1}{2}g^{rr}g_{\varphi\varphi,r'}\Gamma^r_{t\varphi} = -\frac{1}{2}g^{rr}g_{t\varphi,r'}\Gamma^r_{\varphi t} = -\frac{1}{2}g^{rr}g_{t\varphi,r.} \qquad (5)$$

Substituting these expressions in Eq.(3) and cancelling a common factor, we get

$$-\frac{2}{g^{rr}}\frac{d^2r}{ds^2} - g_{tt,r}\left(\frac{dt}{ds}\right)^2 - g_{rr,r}\left(\frac{dr}{ds}\right)^2 + g_{\varphi\varphi,r}\left(\frac{d\varphi}{dr}\right)^2 + 2g_{t\varphi,r}\frac{dt}{ds}\frac{d\varphi}{ds} = 0. \qquad (6)$$

Cancelling ds^2 in the denominators of the terms in Eq.(6), and dividing by $d\varphi^2$ all through in (6), we obtain

$$-\frac{2}{g^{rr}}\frac{d^2r}{d\varphi^2} - g_{rr,r}\left(\frac{dr}{d\varphi}\right)^2 - g_{tt,r}\left(\frac{dt}{d\varphi}\right)^2 + 2g_{t\varphi,r}\frac{dt}{d\varphi} + g_{\varphi\varphi,r} = 0. \qquad (7)$$

For circular orbit, r is not a function ofφ, and hence, we obtain from (7) the following radial geodesic equation in Kerr field:

$$g_{tt,r}\left(\frac{dt}{d\varphi}\right)^2 + 2g_{t\varphi,r}\frac{dt}{d\varphi} + g_{\varphi\varphi,r} = 0. \qquad (8)$$

Solution to this equation is

$$\frac{dt}{d\varphi} = -\frac{g_{t\varphi,r}}{g_{tt,r}} \pm \sqrt{\left(\frac{g_{t\varphi,r}}{g_{tt,r}}\right)^2 - \frac{g_{\varphi\varphi,r}}{g_{tt,r}}}. \qquad (9)$$

Now,

$$g_{t\varphi,r} = \frac{d}{dr}\left(-\frac{2Ma}{r}\right) = \frac{2Ma}{r^2}, \qquad (10)$$

$$g_{tt,r} = -\frac{2M}{r^2}, \qquad (11)$$

$$g_{\varphi\varphi,r=} -\frac{2Ma^2}{r^2} + 2r. \qquad (12)$$

Substitution of these results into Eq.(9), we obtain

$$\frac{dt}{d\varphi} = a \pm \sqrt{\frac{r^3}{M}} \qquad (13)$$

In true units this formula reduces to

$$\frac{dt}{d\varphi} = \frac{a}{c} \pm \sqrt{\frac{r^3}{GM}} = \frac{a}{c} \pm \omega_k^{-1}. \qquad (14)$$

Here, ω_k is the Newtonian angular speed of the test body in its orbit. The gravitomagnetic clock effect rises from the difference in orbital period in prograde and retrograde orbits. Prograde orbit is one where the sense of orbital motion of the test body is same as the sense of the rotation of the central body. In retrograde orbit the situation is opposite. Note that in Eq.(14) the +(-) sign refers to prograde (retrograde) motion. The orbital period in prograde motion can be found by integrating Eq. (14) with the plus sign from 0 to 2π. The orbital period in retrograde orbit can be found by integrating Eq.(14) with the minus sign from 0 to -2π.

We indicate by T_- the prograde period which is

$$T_+ = \int_0^{2\pi}\left(\frac{a}{c} + \frac{1}{\omega_k}\right)d\varphi = 2\pi\frac{a}{c} + \frac{2\pi}{\omega_k}. \qquad (15)$$

Similarly,

$$T_- = \int_0^{-2\pi}\left(\frac{a}{c} - \frac{1}{\omega_k}\right)d\varphi = -2\pi\frac{a}{c} + \frac{2\pi}{\omega_k}. \qquad (16)$$

which is the retrograde orbital period. The period difference is

$$T_+ - T_- = 4\pi\frac{a}{c} \qquad (17)$$

This is the expression for gravitomagnetic clock effect. Clearly, the retrograde orbit is faster than the prograde orbit. The calculation in this section can be found in [11].

2. CLOCK EFFECT DUE TO GRAVITATIONAL SPIN-ORBIT COUPLING

When a non-spinning test particle orbits a Kerr source, there appears the gravitomagnetic clock effect which is the difference between orbital periods of the test particle motion in prograde and retrograde orbits. When a spinning particle orbits a spin-less central object, the case is that of motion of a spinning particle in Schwarzschild metric. The clock effect in this case is also termed as gravitomagnetic clock effect.

For our purpose here, we consider the gravitoelectricfield of the Schwarzschild object as given by Newtonian potential. The Hamiltonian describing motion of a spinning test particle in circular orbit around the non-spinning central body is

$$H = \frac{p^2}{2m} + m\emptyset + \frac{3}{2}\frac{GM}{mc^2r^3}\vec{S}.\vec{L}, \tag{18}$$

Where the spin-orbit coupling term is given equation$\beta\frac{\vec{s}.(\vec{g}\times\vec{p})}{2mc^2} = \frac{\beta}{2m^2c^2r}\vec{S}.\vec{L}\frac{dV_m}{dr}$, but with the coefficient chosen as that reported in [12].

As mentioned earlier, the test particle moves in the equatorial x-y plane with its spin S pointing along the positive z-direction. In this case, the magnitude of the orbital angular momentum of the particle is $l = mr^2\frac{d\varphi}{dt}$, where φ the angle is measuring the angular position of the particle in the x-y plane. To obtain the force on the particle we use Hamilton's equation, $F_i = \dot{p}_i = -\frac{\partial H}{\partial x_i}$. In differentiating the Hamiltonian with respect to position the momentum is kept fixed, as usual. In this way we obtain

$$-mr\left(\frac{d\varphi}{dt}\right)^2\hat{r} = -\frac{GMm}{r^2}\hat{r} + \frac{3GM}{c^2r^2}\left(\frac{d\varphi}{dt}\right)\hat{r} \tag{19}$$

Now Eq.(19) can be reduced to

$$\left(\frac{d\varphi}{dt}\right)^2 + \frac{3GMS}{mc^2r^3}\left(\frac{d\varphi}{dt}\right) - \frac{GM}{r^3} = 0. \tag{20}$$

The quadratic equation (20) has solutions

$$\frac{d\varphi}{dt} = \frac{1}{2}\left(-\frac{3GMS}{mc^2r^3}\varphi \pm \sqrt{\left(\frac{3GMS}{mc^2r^3}\right)^2 + \frac{4GM}{r^3}}\right). \tag{21}$$

Neglecting the first term under the square root, since it is proportional to S^2/c^4, which, in realistic cases, is very small, we obtain

$$\frac{d\varphi}{dt} = -\frac{3GMS}{2mc^2r^3} \pm \sqrt{\frac{GM}{r^3}} \tag{22}$$

The second term in this equation is simply the Keplerian angular speed in circular orbit;$\omega_k = \sqrt{\frac{GM}{r^3}}$. Note that neglecting spin effect, we would have from Eq. (20),

$\left(\frac{d\varphi}{dt}\right)^2 = \frac{GM}{r^3}$, leading to $\omega_k = \sqrt{\frac{GM}{r^3}}$. Here the positive sign refers to increasingφ, i.e., counter clockwise motion. Following this notion, we write from Eq. (22), the two angular frequencies:

$$\left(\frac{d\varphi}{dt}\right)_+ = -\frac{3GMS}{2mc^2r^3} + \omega_k \tag{23}$$

$$\left(\frac{d\varphi}{dt}\right)_- = -\frac{3GMS}{2mc^2r^3} - \omega_k \tag{24}$$

The second term of Eq. (23) and Eq. (24) is larger than the first term, so, in finding$\frac{dt}{d\varphi}$, we use the binomial theorem and obtain the following approximate formula:

$$\left(\frac{dt}{d\varphi}\right)_+ = \frac{1}{\omega_k} + \frac{3S}{2mc^2} \tag{25}$$

and $$\left(\frac{dt}{d\varphi}\right)_- = -\frac{1}{\omega_k} + \frac{3S}{2mc^2} \tag{26}$$

Integrating Eq. (25) from $\varphi = 0\ to\ \varphi = 2\pi$ and integrating Eq. (26) from$\varphi = 0\ to\ \varphi = -2\pi$, we obtain the T_+andT_-, respectively. Hence

$$T_+ = \frac{2\pi}{\omega_k} + \frac{3\pi S}{mc^2} \tag{27}$$

and $$T_- = \frac{2\pi}{\omega_k} - \frac{3\pi S}{mc^2} \tag{28}$$

The period difference is

$$T_+ - T_- = \frac{6\pi S}{mc^2} \tag{29}$$

This is the gravitomagnetic clock effect in spinning test particle orbit around a Schwarzschild source. However, the calculation method employed here is very elementary and very illuminating. One thing to note here is that the contribution of spin of the orbiting test particle is always given by Eq. (29), no matter whether it is in motion round a Schwarzschild source or a Kerr source. Therefore, the source of asymmetry in period of prograde and retrograde motion of a spinning particle is the spin-orbit coupling.

CONCLUSION

In this paper, we have worked out the consequence of gravitational spin-orbit coupling of the spin and orbital angular momentum of a test particle which is in motion round a massive non-spinning body.The effect of gravitational spin-orbit coupling, which we have worked out here, is the gravitomagnetic clock effect. This effect is the asymmetry of time periods of prograde and retrograde motion in the orbit of a non-spinning particle round a Kerr source. However, when the central body is non-spinning, there also appears a similar effect in case the orbiter is a spinning test particle. We show that the gravitomagnetic clock effect in Schwarzschild is given by $6\pi s/mc^2$. However the calculation method employed here is very elementary and very illuminating. One thing to note here is that the contribution of spin of the orbiting test particle is always given by Eq. (29), no matter whether it is in motion round a Schwarzschild or a Kerr source. Therefore, the source of asymmetry in period of prograde and retrograde motion of a spinning particle is the spin-orbit particle.

In conclusion, we have discussed the consequence of gravitational spin-orbit coupling and found important results that confirm previous results and also present new results to be verified.

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